Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Analysis II

Mid-term examination — Date : March 7, 2014
Total Marks: 105 Maximum marks: 100 — Time: 3 hours

In the following a, b are real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a bounded function

1. Show that
$$\underline{\int} f \leq \overline{\int} f.$$
[15]

2. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable then $f^2 : [a, b] \to \mathbb{R}$ is also Riemann integrable. [15]

3. Suppose $f : [a, b] \to \mathbb{R}$ is continuous, show that
$$\int_a^b f = f(c)(b - a)$$
for some c with $a \leq c \leq b$. [15]

4. Define $h : [0, 1] \to \mathbb{R}$ by
$$h(x) = \begin{cases} \frac{1}{n^2} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}; \\ 0 & \text{Otherwise.} \end{cases}$$
Show that h is Riemann integrable. Compute $\int_0^1 h$. [15]

5. Let U be a finite set. Let X be the collection of all subsets of U. Define $d : X \times X \to \mathbb{R}$ by
$$d(A, B) = \sharp(A \bigcup B) - \sharp(A \bigcap B),$$
where $\sharp(S)$ denotes the number of elements in S for any set S. Prove or disprove the claim that d is a metric on X. [15]

[P.T.O.]

6. Consider the space of complex numbers with respect to usual metric: $d(z, w) = |z - w|$ for z, w in $\mathbb{C}$. Identify interior, exterior and boundary for the following subsets of $\mathbb{C}$.

$$(i) \quad A = \{z \in \mathbb{C} : 2 \leq |z| < 3\};$$

$$(ii) \quad B = \{z \in \mathbb{C} : z + \bar{z} = 0\};$$

$$(iii) \quad C = \{z \in \mathbb{C} : 4 \leq (z + \bar{z}) \leq 5\}.$$

(Here $\bar{z}$ denotes the complex conjugate of z.) [15]

7. Let (Y, d) be a metric space and let C be a non-empty subset of Y. Define a function $g : Y \to \mathbb{R}$ by

$$g(y) = \inf\{d(y, x) : x \in C\}.$$

Show that g is a continuous function on Y. [15]